

Arc diagram varieties

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Abstract: Let k be an algebraically closed field and α, β, γ be partitions. An algebraic group acts on the constructible set of short exact sequences of nilpotent k -linear operators of Jordan types α, β , and γ , respectively; we are interested in the stratification given by the orbits in the case where all parts of α are at most 2. Geometric properties of the degeneration relation are controlled by the combinatorics of arc diagrams. The extended bubble sort algorithm is used to construct chains of orbits such that subsequent strata have dimension difference equal to one.

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1. Introduction

Let k be an algebraically closed field. For a partition $\alpha = (\alpha_1 \geq \dots \geq \alpha_n)$ we denote by N_α the nilpotent linear operator $T : V \rightarrow V$ where V is a k -vector space of dimension $|\alpha| = \alpha_1 + \dots + \alpha_n$ and where the operator T can be represented by a matrix of Jordan type α . Denote by \mathcal{N} the category of all nilpotent linear operators. It is well-known that the map $\alpha \mapsto N_\alpha$ defines a one-to-one correspondence between the set of all partitions and the set of isomorphism classes of objects in \mathcal{N} [14, II,(1.4)].

Let α, β, γ be partitions. The affine variety $\mathbb{H}_\alpha^\beta = \text{Hom}_k(N_\alpha, N_\beta)$ (consisting of all $|\beta| \times |\alpha|$ -matrices with coefficients in k) with the Zariski topology contains as constructible subset the set $\mathbb{V}_{\alpha,\gamma}^\beta$ of monomorphisms $f : N_\alpha \rightarrow N_\beta$ such that $\text{Coker } f \cong N_\gamma$. We consider $\mathbb{V}_{\alpha,\gamma}^\beta$ as a topological space with the induced topology.

On $\mathbb{V}_{\alpha,\gamma}^\beta$ acts the algebraic group $G = \text{Aut}_{\mathcal{N}}(N_\alpha) \times \text{Aut}_{\mathcal{N}}(N_\beta)$ via $(g, h) \cdot f = hfg^{-1}$. The orbits of this action correspond bijectively to the isomorphism classes of short exact sequences

$$0 \longrightarrow N_\alpha \longrightarrow N_\beta \longrightarrow N_\gamma \longrightarrow 0$$

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of the given type (α, β, γ) . In the case where all parts of α are at most 2, the orbits are in one-to-one correspondence with arc diagrams and define a stratification for $\mathbb{V}_{\alpha, \gamma}^\beta$.

In Section 2 we compute the orbit dimensions, and describe in terms of operations on arc diagrams which orbits form the boundary of a given orbit. The orbits together with the degeneration relation form the partially ordered set $\mathcal{D}_{\alpha, \gamma}^\beta$. We review results from [11] and list some references regarding the history of the underlying counting and isomorphism problems for subgroup embeddings.

In Section 3 we deal with the question whether all saturated chains in $\mathcal{D}_{\alpha, \gamma}^\beta$ have the same length. While this is not the case in general, we obtain a positive answer in case $\beta \setminus \gamma$ is a horizontal stripe (Corollary 3.7).

In this situation, the extended bubble sort algorithm in Section 4 produces saturated chains in $\mathcal{D}_{\alpha, \gamma}^\beta$ such that any two subsequent orbits have dimension difference one.

In the last Section 5 we discuss links to projective varieties; in fact, projective spaces and Grassmannians occur as epimorphic images of arc diagram varieties of type $\mathbb{V}_{\alpha, \gamma}^\beta$. Finally we note that the degeneration order for nilpotent operators is just the opposite order of a natural partial ordering for Littlewood-Richardson tableaux (Proposition 5.4).

2. The stratification

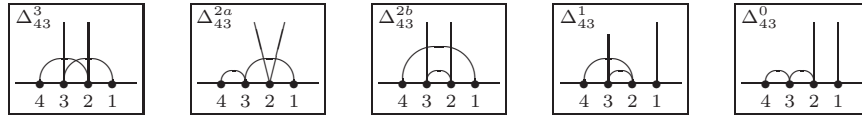
We assume throughout that k is an algebraically closed field, and that α, β, γ are partitions where α is such that all parts in α are at most 2, i.e. $\alpha_1 \leq 2$ holds. Then the conjugate α' of α has two parts $\alpha' = (\alpha'_1, \alpha'_2)$ where α'_2 counts the number of 2's in α and $\alpha'_1 - \alpha'_2$ the number of 1's.

2.1. From short exact sequences to arc diagrams

Definition: 1. An **arc diagram Δ of type (α, β, γ)** has α'_2 arcs and $\alpha'_1 - \alpha'_2$ poles which are arranged such that at each point i , the number of arcs and poles starting or ending is $\beta'_i - \gamma'_i$.

2. By $\mathcal{D}_{\alpha, \gamma}^\beta$ we denote the set of all arc diagrams of type (α, β, γ) .

Example: Let $\alpha = (2, 2, 1, 1)$, (so $\alpha' = (4, 2)$), $\beta = (4, 3, 3, 2, 2, 1)$, $\gamma = (3, 2, 2, 1, 1)$. The following arc diagrams have type (α, β, γ) .



Before we give a detailed description of the stratification $\{\mathbb{V}_\Delta : \Delta \in \mathcal{D}_{\alpha, \gamma}^\beta\}$ for $\mathbb{V}_{\alpha, \gamma}^\beta$, we review briefly how tableaux provide a link between short exact sequences and arc diagrams.

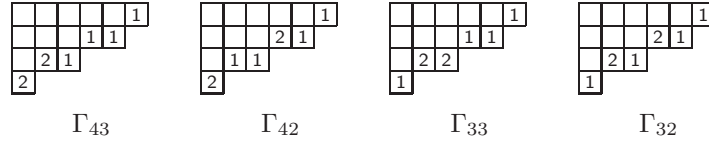
The following result is stated in [8] for p -groups:

THEOREM 2.1. *Given partitions α, β, γ , there exists a short exact sequence of nilpotent linear operators $0 \rightarrow N_\alpha \rightarrow N_\beta \rightarrow N_\gamma \rightarrow 0$ if and only if there exists a Littlewood-Richardson (LR-) tableau Γ of type (α, β, γ) .*

Definition: 1. Given three partitions α, β, γ , an **LR-tableau** of type (α, β, γ) is a skew diagram of shape $\beta \setminus \gamma$ with α'_1 entries $\boxed{1}$, α'_2 entries $\boxed{2}$, etc. The entries are weakly increasing in each row, strictly increasing in each column, and satisfy the lattice permutation property (for each $c \geq 0$, $\ell \geq 2$ there are at least as many entries $\ell - 1$ on the right hand side of the c -th column as there are entries ℓ).

2. The **LR-coefficient** $c_{\alpha, \gamma}^\beta$ counts the number of LR-tableaux of type (α, β, γ) .

Example: Let $\alpha = (2, 2, 1, 1)$, $\beta = (4, 3, 3, 2, 2, 1)$, $\gamma = (3, 2, 2, 1, 1)$. There are 4 LR-tableaux of type (α, β, γ) , so $c_{\alpha, \gamma}^\beta = 4$.



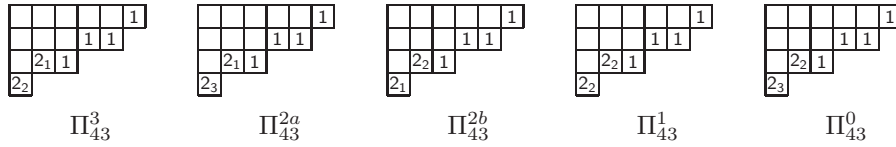
(In the expression Γ_{ij} , the subscript ij lists the rows which contain the symbol $\boxed{2}$, and hence determines the LR-tableau uniquely in the case where $\alpha_1 \leq 2$.)

Definition: 1. A **Klein tableau** of type (α, β, γ) is a refinement of the LR-tableau of the same type in the sense that each entry $\ell \geq 2$ carries a subscript, subject to the following conditions (see [9, (1.2)]):

- (a) If a symbol $\boxed{\ell_r}$ occurs in the m -th row in the tableau, then $1 \leq r \leq m - 1$.
- (b) If $\boxed{\ell_r}$ occurs in the m -th row and the entry above $\boxed{\ell_r}$ is $\ell - 1$, then $r = m - 1$.
- (c) The total number of symbols $\boxed{\ell_r}$ in the tableau cannot exceed the number of entries $\ell - 1$ in row r .

2. Let Γ be an LR-tableau with entries at most 2, and Π a Klein tableau which refines Γ . The **arc diagram corresponding to Π** is obtained by drawing an arc from m to j for each pair of boxes $\boxed{2_j}$ in row m and $\boxed{1}$ in row j , and by drawing a pole at r for each remaining box $\boxed{1}$ in row r .

Example: Here are the five Klein tableaux which refine the LR-tableau Γ_{43} .



The arc diagrams $\Delta_{43}^3, \Delta_{43}^{2a}, \Delta_{43}^{2b}, \Delta_{43}^1$ and Δ_{43}^0 given by the above Klein tableaux are pictured in the beginning of this section. (The exponent x in Δ_{ij}^x counts the number of intersections.)

By \mathcal{S}_2 we denote the category of all sequences $0 \rightarrow N_\alpha \rightarrow N_\beta \rightarrow N_\gamma \rightarrow 0$ where α has all parts at most 2. Based on the classification of the indecomposable embeddings in [1, Theorem 7.5], it is shown in [20, Proposition 2] that there is a one-to-one correspondence

$$\{\text{objects in } \mathcal{S}_2\} / \cong \xleftrightarrow{1-1} \{\text{Klein tableaux with entries at most 2}\}.$$

Summarizing we obtain:

COROLLARY 2.2. *There is a one-to-one correspondence*

$$\mathcal{D}_{\alpha,\gamma}^\beta \xleftrightarrow{1-1} \{G\text{-orbits in } \mathbb{V}_{\alpha,\gamma}^\beta\}.$$

2.2. Strata given by arc diagrams

Definition: For an arc diagram Δ of type (α, β, γ) we denote the corresponding G -orbit in $\mathbb{V}_{\alpha,\gamma}^\beta$ by \mathbb{V}_Δ .

PROPOSITION 2.3. *The set $\{\mathbb{V}_\Delta : \Delta \in \mathcal{D}_{\alpha,\gamma}^\beta\}$ forms a stratification for $\mathbb{V} = \mathbb{V}_{\alpha,\gamma}^\beta$ in the sense that*

1. *Each \mathbb{V}_Δ is locally closed in \mathbb{V} .*
2. *\mathbb{V} is the disjoint union $\bigcup_{\Delta \in \mathcal{D}_{\alpha,\gamma}^\beta}^\bullet \mathbb{V}_\Delta$.*
3. *For each Δ there is a finite subset $U_\Delta \subset \mathcal{D}_{\alpha,\gamma}^\beta$ such that the closure $\overline{\mathbb{V}}_\Delta$ is just the union $\bigcup_{\Gamma \in U_\Delta} \mathbb{V}_\Gamma$.*

Proof. We have seen in Lemma 2.2 that $\mathbb{V}_{\alpha,\gamma}^\beta$ is the finite union of the G -orbits of type \mathbb{V}_Δ . According to [7, Proposition 8.3], each orbit is a smooth and locally closed subset of $\mathbb{V}_{\alpha,\gamma}^\beta$ whose boundary is a union of orbits of strictly lower dimension. \square

Remark: 1. The condition on the field k to be algebraically closed is only needed for the last statement. Otherwise, the field can be arbitrary, in fact, there need not even be a field: For Λ a discrete valuation domain with maximal ideal m , we can define $N_\alpha(\Lambda) = \bigoplus_{i=1}^s \Lambda / (m^{\alpha_i})$. In particular if Λ is the localization \mathbb{Z}_p , then we are dealing with finite abelian p -groups.

2. The problem of classifying the orbits in $\mathbb{V}_{\alpha,\gamma}^\beta$ has been posed by G. Birkhoff in 1934 [2] for $\Lambda = \mathbb{Z}_p$: Classify all subgroups A of a finite abelian p -group B , up to automorphisms of B . In general, the problem is considered infeasible, see for example [18], but there are many partial and related results: If the exponent of B is at most 5, then the category of embeddings has finite type [16]; for $\Lambda = k[T]_{(T)}$, tame type occurs if the exponent of B

is at most 6 [17]; our category \mathcal{S}_2 has discrete representation type [1]; for the related problem of studying lattices over tiled orders we refer to [19]; categories of embeddings of graded operators occur in singularity theory [13]; for a classification of the representation types of chain categories we refer to [21]; please see [22] for homological properties of categories of embeddings.

2.3. The dimensions of the strata

In this subsection we review the dimension formula:

PROPOSITION 2.4. *Let Δ be an arc diagram of type (α, β, γ) . The stratum \mathbb{V}_Δ is a smooth irreducible variety of dimension*

$$\dim \mathbb{V}_\Delta = \deg g_{\alpha, \gamma}^\beta + \deg a_\alpha - x(\Delta).$$

First we define the terms in the dimension formula. From results given in [9] one can deduce the following theorem.

THEOREM 2.5. *For any partition triple (α, β, γ) there exists a polynomial $g_{\alpha, \gamma}^\beta(t) \in \mathbb{Z}[t]$ such that for any finite field k we have*

$$g_{\alpha, \gamma}^\beta(|k|) = |\mathbb{V}_{\alpha, \gamma}^\beta(k)|,$$

where $|X|$ denotes the cardinality of the finite set X .

Polynomials $g_{\alpha, \gamma}^\beta(t)$ are called **Hall polynomials**. It is known (see [14]) that

$$\deg g_{\alpha, \gamma}^\beta(t) = n(\beta) - n(\alpha) - n(\gamma),$$

where for a partition λ the moment is defined as

$$n(\lambda) = \sum_{i \geq 0} (i-1)\lambda_i.$$

A formula for the cardinality $a_\alpha(q) = |\text{Aut } N_\alpha(\mathbb{F}_q)|$ of the automorphism group of N_α is in [14, II, (1.6)]. In particular, $\deg a_\alpha = |\alpha| + 2n(\alpha)$.

For an arc diagram Δ , we denote by $x(\Delta)$ the number of intersections in Δ .

Definition: For a Littlewood-Richardson tableau Γ of type (α, β, γ) , we say an arc diagram Δ has **Littlewood-Richardson type** Γ if for each i , the number of arcs in Δ starting at i equals the number of 1's in the i -th row of Γ . We write $\mathbb{V}_\Gamma = \bigcup_{\Delta \text{ has type } \Gamma} \mathbb{V}_\Delta$.

It follows from the previous section:

$$\mathbb{V}_{\alpha, \gamma}^\beta = \bigcup_{\Gamma} \mathbb{V}_\Gamma = \bigcup_{\Gamma} \bigcup_{\Delta} \mathbb{V}_\Delta, \quad (2.6)$$

where the first union is indexed by all LR-tableaux Γ of type (α, β, γ) and the second union is indexed by all arc diagrams Δ of type Γ .

It is well known that orbits of an algebraic group action are locally closed sets. It follows that $\mathbb{V}_{\alpha, \gamma}^{\beta}(k)$ and $\mathbb{V}_{\Gamma}(k)$ are constructible sets, because they are finite unions of locally closed sets $\mathbb{V}_{\Delta}(k)$.

Correspondingly, if k is a finite field of q elements, there is the following sum formula for Hall polynomials.

$$g_{\alpha, \gamma}^{\beta}(q) = \sum_{\Gamma} g_{\Gamma}(q) = \sum_{\Gamma} \sum_{\Delta} g_{\Delta}(q), \quad (2.7)$$

where the indices are as above. The polynomials g_{Γ} are monic of the same degree $n(\beta) - n(\alpha) - n(\gamma)$, while the polynomials g_{Δ} are monic of degree $n(\beta) - n(\alpha) - n(\gamma) - x(\Delta)$ ([9, Corollaries 1-3], here $x(\Delta)$ is the deviation from dominance of the prototype given by the arc diagram Δ).

The formulae (2.6) and (2.7) have a different nature: the first one is geometric and the second one is combinatorial. The following remarks show that they are "compatible".

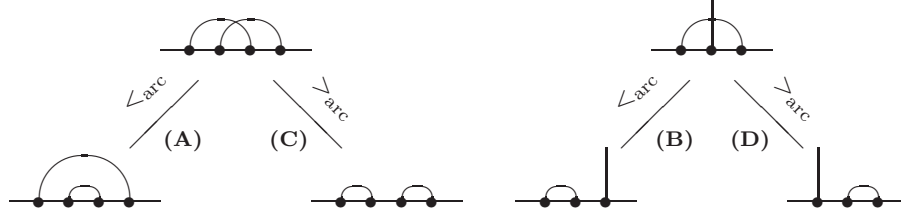
Results presented in [11, Section 5] give us the following formulae for variety dimensions.

- $\dim \mathbb{V}_{\alpha, \gamma}^{\beta} = \deg g_{\alpha, \gamma}^{\beta} + \deg a_{\alpha},$
- $\dim \mathbb{V}_{\Gamma} = \deg g_{\Gamma} + \deg a_{\alpha} = \dim \mathbb{V}_{\alpha, \gamma}^{\beta},$
- $\dim \mathbb{V}_{\Delta} = \deg g_{\Delta} + \deg a_{\alpha} = \dim \mathbb{V}_{\alpha, \gamma}^{\beta} - x(\Delta)$

Remark: Polynomials and algebras, that we call Hall polynomials and Hall algebras, were defined and investigated in 1900 by E. Steinitz. He described their connections with Schur functions. However results of Steinitz were forgotten. In 50thies of XX century Hall polynomials and algebras were defined by P. Hall for finite abelian p -groups. In [5], P. Hall gave only summary of this theory. His work continued J. A. Green [4] and T. Klein [9]. The reader is referred to [14] for more informations about Hall polynomials and algebras and their connections with symmetric functions.

2.4. Geometric properties of $\mathbb{V}_{\alpha, \gamma}^{\beta}$

Definition: Two diagrams of arcs and poles are said to be in **arc order** if the first is obtained from the second by a sequence of moves of type



If the arc diagrams Δ and Δ' are in relation, we write $\Delta \leq_{\text{arc}} \Delta'$.

The main result in [11] states that the arc order and the degeneration order on arc diagrams are related:

THEOREM 2.8. *Suppose that k is an algebraically closed field and that α, β, γ are partitions with $\alpha_1 \leq 2$. For arc diagrams Δ, Δ' of type (α, β, γ) we have*

$$\Delta \leq_{\text{deg}} \Delta' \quad \text{if and only if} \quad \Delta \leq_{\text{arc}} \Delta'$$

where by definition $\Delta \leq_{\text{deg}} \Delta'$ if and only if $\mathbb{V}_{\Delta'} \subseteq \overline{\mathbb{V}}_{\Delta}$.

The Littlewood-Richardson coefficient $c_{\alpha, \gamma}^{\beta}$ counts the number of LR-tableaux Γ of type (α, β, γ) , see [14]. It follows that in the sum (2.7) there exist exactly $c_{\alpha, \gamma}^{\beta}$ polynomials g_{Γ} of degree $n_{\beta} - n_{\alpha} - n_{\gamma}$.

Geometrically it means that there exist $c_{\alpha, \gamma}^{\beta}$ subsets $\mathbb{V}_{\Gamma} \subseteq \mathbb{V}_{\alpha, \gamma}^{\beta}$ with the maximal dimension $n_{\beta} - n_{\alpha} - n_{\gamma} + \deg a_{\alpha}$. Moreover, for any such a subset \mathbb{V}_{Γ} we have

$$\dim \mathbb{V}_{\Gamma} = n_{\beta} - n_{\alpha} - n_{\gamma} + \deg a_{\alpha} = \dim \mathbb{V}_{\alpha, \gamma}^{\beta}$$

and $\mathbb{V}_{\alpha, \gamma}^{\beta} = \bigcup_{\Gamma} \overline{\mathbb{V}}_{\Gamma}$, where union runs over all LR-tableaux Γ with maximal dimension. As a consequence we get the following fact.

LEMMA 2.9. 1. $\mathbb{V}_{\alpha, \gamma}^{\beta}$ is irreducible if and only if $c_{\alpha, \gamma}^{\beta} = 1$.

2. There exists $c_{\alpha, \gamma}^{\beta}$ irreducible components of $\mathbb{V}_{\alpha, \gamma}^{\beta}$.

For any LR-tableau Γ there exists exactly one arc diagram with no intersections, see [11]. This diagram Δ satisfies $\overline{\mathbb{V}}_{\Delta} \cap \mathbb{V}_{\Gamma} = \mathbb{V}_{\Gamma}$. We deduce the following fact.

LEMMA 2.10. \mathbb{V}_{Γ} is an irreducible set.

Moreover $\mathbb{V}_{\alpha, \gamma}^{\beta} = \bigcup_{\Delta} \overline{\mathbb{V}}_{\Delta}$, where the union runs over all arc diagrams with no intersections.

3. Partially ordered sets

Let \mathcal{D}_Γ be the set of all arc diagrams given by an LR-tableau Γ with entries at most two, and $\mathcal{D}_{\alpha,\gamma}^\beta$ the set of all arc diagrams of partition type (α, β, γ) with $\alpha_1 \leq 2$. We describe properties of the posets $\mathcal{D}_\Gamma = (\mathcal{D}_\Gamma, \leq_{\text{arc}})$ and $\mathcal{D}_{\alpha,\gamma}^\beta = (\mathcal{D}_{\alpha,\gamma}^\beta, \leq_{\text{arc}})$.

In [11] the following theorem is proved.

THEOREM 3.1. *Let Γ be an LR tableau with entries at most two.*

1. *In the poset \mathcal{D}_Γ there exists exactly one minimal element: the arc diagram with no intersections.*
2. *In the poset \mathcal{D}_Γ there exists exactly one maximal element, the arc diagram with the maximal number of intersections.*
3. *In the poset $\mathcal{D}_{\alpha,\gamma}^\beta$ there exists exactly one maximal element, given by the unique arc diagram with the largest number of intersections.*
4. *The set of minimal elements of the poset $\mathcal{D}_{\alpha,\gamma}^\beta$ consists of the intersection-free arc diagrams, they are in one-to-one correspondence with the LR-tableaux of type (α, β, γ) .*

COROLLARY 3.2. 1. *The open strata in $\{\mathbb{V}_\Delta : \Delta \in \mathcal{D}_{\alpha,\gamma}^\beta\}$ are the strata of maximal dimension. The number of such strata is $c_{\alpha,\gamma}^\beta$, they are in one-to-one correspondence with the intersection-free arc diagrams.*

2. *There is a unique closed stratum, it has minimal dimension and is given by the unique arc diagram with the maximal number of intersections.*

By identifying the points in $\mathcal{D}_{\alpha,\gamma}^\beta$ which correspond to the same LR-tableau, we obtain the coarser poset $\overline{\mathcal{D}}_{\alpha,\gamma}^\beta$ on the set of LR-tableaux of type (α, β, γ) . Proposition 5.4 below shows that there are several equivalent candidates for a partial ordering on $\overline{\mathcal{D}}_{\alpha,\gamma}^\beta$.

3.1. Two questions about saturated chains

Definition: A chain in a poset is **saturated** if it has no refinement.

Question: In $\mathcal{D}_{\alpha,\gamma}^\beta$, do all saturated chains have the same length?

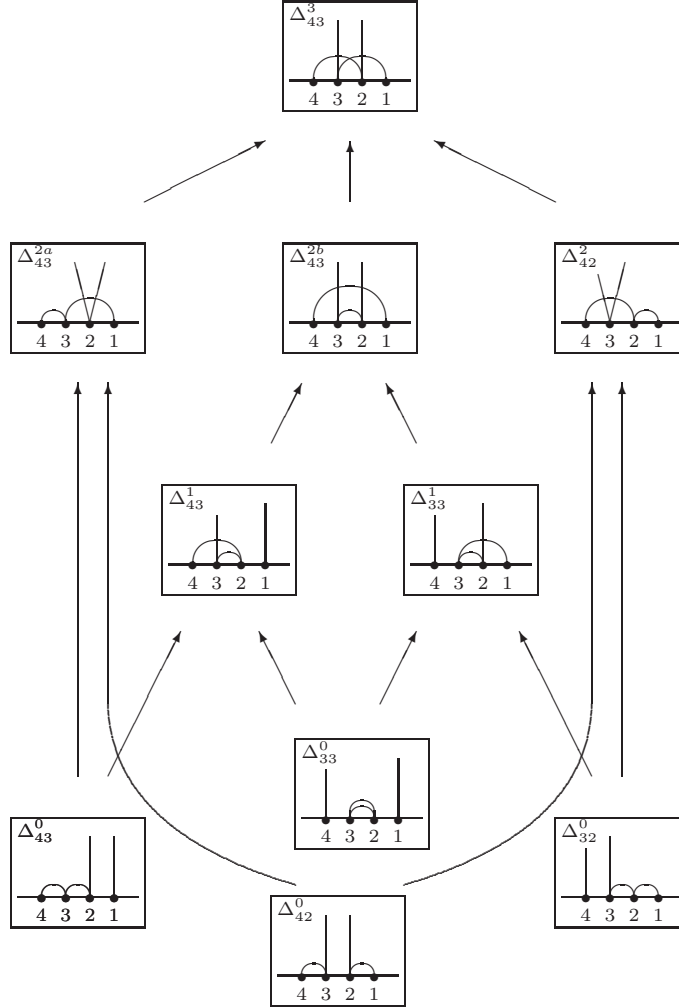
Question: Suppose $\Delta < \Delta'$ in $\mathcal{D}_{\alpha,\gamma}^\beta$. Is there a chain from Δ to Δ' where subsequent strata have dimension difference equal to one?

The example below shows that the answer to both questions is NO. Take $\alpha = (2, 2, 1, 1)$, $\beta = (4, 3, 3, 2, 2, 1)$, and $\gamma = (3, 2, 2, 1, 1)$.

Consider the five arc diagrams labeled Δ_{43}^x . There are two saturated chains from Δ_{43}^0 to Δ_{43}^3 , they have length 2 and 3, respectively. Similarly, there are two saturated chains from Δ_{32}^0 to Δ_{43}^3 , also of length 2 and 3, respectively.

Note that for each direct predecessor of Δ_{43}^{2a} and Δ_{42}^2 , the dimension of the corresponding stratum decreases by two.

In the next sections we will obtain an affirmative answer to both questions in the case where $\beta \setminus \gamma$ is a vertical strip (which excludes double poles in any of the arc diagrams).



Example: Hasse diagram for $\mathcal{D}_{\alpha, \gamma}^{\beta}$ where $\alpha = (2211)$, $\beta = (433221)$, $\gamma = (32211)$

3.2. Sequences of sources and targets

Formally, an arc diagram is a finite set of arcs and poles in the Poincaré half plane. We assume that all end points are natural numbers (arranged from right to left) and permit multiple arcs and poles.

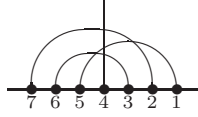
We call the left end of an arc the **source** and the right end the **target**.

We view a pole in an arc diagram as an arc with the source equals to ∞ and the target equals to the end point of the pole.

With an arc diagram we associate a chain of pairs of numbers in the following way.

Starting from the left side of the diagram if we meet a target of an arc f we write (m, n) , where n is the target of f and m is the source of f . If two or more arcs have the same target n we arrange corresponding pairs $(m_1, n), \dots, (m_x, n)$ in such a way that $m_1 \leq m_2 \leq \dots \leq m_x$.

Example: With the following diagram



we associate the chain:

$$(\infty, 4), (6, 3), (7, 2), (5, 1).$$

LEMMA 3.3. *Two arcs f and g , with the corresponding pairs (m, n) and (k, r) of sources and targets, have a crossing if and only if $r < n < k < m$.*

Proof. Straightforward. \square

COROLLARY 3.4. *Let Δ be an arc diagram with the corresponding sequence of sources and targets:*

$$(m_1, n_1), \dots, (m_x, n_x).$$

If $m_i > n_j$, for all $i, j \in \{1, \dots, x\}$, then the following conditions are equivalent

1. Δ is dominant, i.e. there is no crossings in Δ ;
2. $m_1 \leq m_2 \leq \dots \leq m_x$.

3.3. The Bruhat order

Fix an LR-tableau Γ of type (α, β, γ) with $\alpha_1 \leq 2$. Let x denote the number of the boxes with the entry 1. Assume that Γ satisfies the following conditions:

1. The number of boxes with the entry 2 is equal to x or to $x - 1$, i.e. there is at most one pole in the corresponding arc diagram.

2. In any row of Γ there is at most one non-empty box, i.e. at each point in the corresponding arc diagram, there is at most one arc or pole.
3. If j is the number of a row with entry 2 and i is the number of a row with entry 1, then $j > i$, i.e. each starting point of an arc is on the left of every end point of an arc or pole.

We prove that in this case the poset \mathcal{D}_Γ is related to the Bruhat order of a symmetric group.

LEMMA 3.5. *Let Γ be an LR-tableau satisfying the conditions 1,2,3. There is a bijection between the set \mathcal{D}_Γ and the set S_x of all permutations of x elements.*

Proof. Let $\Delta \in \mathcal{D}_\Gamma$. Since Γ satisfies the condition 3, the corresponding sequence of sources and targets of Δ :

$$(m_1, n_1), \dots, (m_x, n_x)$$

is such that $m_i > n_j$, for all $i, j \in \{1, \dots, x\}$. Moreover, the numbers $m_1, n_1, \dots, m_x, n_x$ are pairwise different, because Γ satisfies the conditions 1,2.

With the sequence

$$(m_1, n_1), \dots, (m_x, n_x)$$

we associate the permutation

$$(m_1, m_2, \dots, m_x) = \begin{pmatrix} m_{\sigma(1)} & m_{\sigma(2)} & \dots & m_{\sigma(x)} \\ m_1 & m_2 & \dots & m_x \end{pmatrix},$$

where σ is a permutation such that $m_{\sigma(1)} < m_{\sigma(2)} < \dots < m_{\sigma(x)}$. It is easy to see that this association defines the required bijection. \square

THEOREM 3.6. *Let Γ be an LR-tableau satisfying the conditions 1,2,3. The poset $(\mathcal{D}_\Gamma, \leq_{\text{arc}})$ is isomorphic to the Bruhat order on S_x .*

Proof. Let Δ be an arc diagram with the corresponding sequence of sources and targets:

$$(m_1, n_1), \dots, (m_x, n_x).$$

Note that we can do the move (A) or (B) if and only if there exists a permutation (m_i, m_j) that is an inversion (i.e. $i < j$ but $m_i > m_j$). Therefore the moves of types (A) and (B) in the arc diagram correspond (under the bijection described in the proof of Lemma 3.5) to the inversions in S_x . By [3, Definition 7.16], the Bruhat order on S_x is generated by inversions. We are done. \square

COROLLARY 3.7. *Let Γ be an LR-tableau satisfying the conditions 1,2,3. In the poset \mathcal{D}_Γ all saturated chains have the same length.*

Proof. By [3, Proposition 7.18], in the Bruhat order on S_x all saturated chains have the same length. Therefore, by Theorem 3.6, we are done. \square

3.4. Saturated chains

Let (α, β, γ) be a triple of partitions and let Γ be an LR-tableau of type (α, β, γ) . Let Δ, Δ' be elements of the poset \mathcal{D}_Γ or of the poset $\mathcal{D}_{\alpha, \gamma}^\beta$. We write

- $\Delta <_{\text{arc}} \Delta'$, if $\Delta \neq \Delta'$ and $\Delta \leq_{\text{arc}} \Delta'$,
- $\Delta \rightarrow \Delta'$, if $\Delta <_{\text{arc}} \Delta'$ and there is no Δ'' such that $\Delta <_{\text{arc}} \Delta'' <_{\text{arc}} \Delta'$.

LEMMA 3.8. *Let Γ be an LR-tableau such that in any row of Γ there is at most one non-empty box. Let $\Delta, \Delta' \in \mathcal{D}_\Gamma$. Then $\Delta' \rightarrow \Delta$ if and only if Δ' is obtained from Δ by a single move of type (A) or (B) that reduces the number of crossings by one in the corresponding arc-diagrams.*

Proof. If Δ' is obtained from Δ by a single move of type (A) or (B) that reduces the number of crossings by one in the corresponding arc-diagrams, then obviously $\Delta' \rightarrow \Delta$.

Assume that $\Delta' \rightarrow \Delta$. It is obvious that Δ' is obtained from Δ by a single move of type (A) or (B) that change $(m_i, n_i), (m_j, n_j)$ into $(m_j, n_i), (m_i, n_j)$, where $n_j < n_i < m_j < m_i$. Let Δ and Δ' be the arc-diagrams associated with Δ and Δ' , respectively. Assume that this move reduces the number of crossings by more than one. It follows that in Δ there exists an arc (m_k, n_k) such that the arcs $(m_i, n_i), (m_j, n_j)$ and (m_k, n_k) in Δ have at least 2 more crossings than the arcs $(m_i, n_j), (m_j, n_i)$ and (m_k, n_k) in Δ' . Assume that (m_k, n_k) and (m_i, n_i) have a crossing in Δ . We consider two cases: $n_k < n_i < m_k < m_i$ and $n_i < n_k < m_i < m_k$.

If $n_k < n_i < m_k < m_i$, then we have the following possibilities:

- $n_k < n_j < n_i < m_k < m_j < m_i$, there are 3 crossings in Δ and 2 crossings in Δ' ;
- $n_k < n_j < n_i < m_j < m_k < m_i$, there are 2 crossings in Δ and 1 crossings in Δ' ;
- $n_j < n_k < n_i < m_k < m_j < m_i$, there are 2 crossings in Δ and 1 crossings in Δ' ;
- $n_j < n_k < n_i < m_j < m_k < m_i$. there are 3 crossings in Δ and no crossing in Δ' .

Note that only in the last case the number of crossings is reduced by more than one. But in this case we can obtain Δ' from Δ by the following sequence of moves: in Δ we have: $(m_i, n_i), (m_k, n_k), (m_j, n_j)$; we resolve the crossing $(m_k, n_k), (m_j, n_j)$ and get $(m_i, n_i), (m_j, n_k), (m_k, n_j)$; then we resolve the crossing $(m_i, n_i), (m_j, n_k)$ and get $(m_j, n_i), (m_i, n_k), (m_k, n_j)$; finally resolving the last crossing $(m_i, n_k), (m_k, n_j)$ we get $(m_j, n_i), (m_k, n_k), (m_i, n_j)$ in Δ' . It contradicts the assumption that $\Delta' \rightarrow \Delta$.

In the remaining cases the proof is analogous. \square

COROLLARY 3.9. *Let Γ be an LR-tableau such that in any row of Γ there is at most one non-empty box. In the poset \mathcal{D}_Γ all saturated chains have the same length.*

Proof. From Lemma 3.8 it follows that any saturated chain has the length equal to the number of crossings in the unique arc-maximal diagram in \mathcal{D}_Γ . \square

LEMMA 3.10. *Let (α, β, γ) be a triple of partitions such that $\beta'_i \leq \gamma'_i + 1$ for all i , i.e. $\beta \setminus \gamma$ is a vertical strip; equivalently, there is at most one entry in each row.*

Let $\Delta, \Delta' \in \mathcal{D}_{\alpha, \gamma}^\beta$. Then $\Delta' \rightarrow \Delta$ if and only if Δ' is obtained from Δ by a single move of type (A), (B), (C) or (D) that reduces the number of crossings by one in the corresponding arc-diagrams.

Proof. The proof is analogous to that of Lemma 3.8. \square

COROLLARY 3.11. *Let (α, β, γ) be a triple of partitions such that $\beta'_i \leq \gamma'_i + 1$ for all i . In the poset $\mathcal{D}_{\alpha, \gamma}^\beta$ all saturated chains have the same length.*

Proof. From Lemmata 3.8 and 3.10 it follows that any saturated chain has the length equal to the number of crossings in the unique arc-maximal diagram in $\mathcal{D}_{\alpha, \gamma}^\beta$. \square

Remark: Note that in the poset $\mathcal{D}_{2211, 32211}^{433221}$ given in Example 3.1 there exist saturated chains of different length. Therefore the assumption $\beta'_i \leq \gamma'_i + 1$ for all i , in Corollary 3.11 is necessary. Moreover the diagrams $\Delta_{43}^0, \Delta_{43}^1, \Delta_{43}^{2a}, \Delta_{43}^{2b}$

and Δ_{43}^3 form the poset \mathcal{D}_Γ for $\Gamma = \begin{array}{|c|c|c|c|c|} \hline & & & & 1 \\ \hline & & & 1 & 1 \\ \hline & 2 & 1 & & \\ \hline 2 & & & & \\ \hline \end{array} = \Gamma_{43}$. Note that in this poset

there exist saturated chains having different length. Therefore the assumption that in any row of Γ there is at most one non-empty box, given in Corollary 3.9, is necessary.

4. An algorithmic approach

We describe an algorithm that using moves of type (A) and (B) transforms arbitrary arc diagram to the dominant one. We identify arc diagrams with the corresponding sequences of sources and targets.

4.1. The bubble sort

Algorithm: **The classical bubble sort.**

Input: A sequence of sources and targets:

$$(m_1, n_1), \dots, (m_x, n_x).$$

Output: The sequence

$$(m_{\sigma(1)}, n_1), \dots, (m_{\sigma(x)}, n_x),$$

where $m_{\sigma(1)} \leq m_{\sigma(2)} \leq \dots \leq m_{\sigma(x)}$ and σ is a permutation of the set $\{1, \dots, x\}$.

Description of the algorithm:

1. $y := x$
 2. repeat
 - (a) for all $i = 1, \dots, y - 1$ do
 - (i) if $m_{i+1} < m_i$, then exchange sources m_i with m_{i+1} , i.e. instead of $(m_i, n_i), (m_{i+1}, n_{i+1})$ we get $(m_{i+1}, n_i), (m_i, n_{i+1})$ (we do the move of type (A) or (B) in the corresponding arc-diagram)
 - (b) $y := y - 1$
- until $y = 1$.

LEMMA 4.1. *Let Δ be an arc diagram with the corresponding sequence of sources and targets*

$$(m_1, n_1), \dots, (m_x, n_x)$$

such that $m_i > n_j$, for all $i, j \in \{1, \dots, x\}$. If in Δ there is no multiple poles, then every move in (i) in the algorithm reduces the number of crossings by one.

Proof. If we have a crossing of the arcs (m_i, n_i) and (m_{i+1}, n_{i+1}) , then $m_i > m_{i+1}$. After the exchange in (i) we get a new diagram Δ' with the arcs (m_{i+1}, n_i) and (m_i, n_{i+1}) with no crossing.

If there is a crossing of arcs (m_i, n_i) and (m_k, n_k) in Δ , where $k \neq i + 1$, then $n_k < n_i < m_k < m_i$ or $n_i < n_k < m_i < m_k$. Consider the case $n_k < n_i < m_k < m_i$. Note that $n_k < n_{i+1} < n_i < m_k < m_i$ and therefore $n_k < n_{i+1} < m_k < m_i$. It follows that there is a crossing of (m_k, n_k) and (m_i, n_{i+1}) in Δ' . In the case $n_i < n_k < m_i < m_k$, we have $n_{i+1} < n_i < n_k < m_i < m_k$ and a crossing of (m_k, n_k) and (m_i, n_{i+1}) in Δ' .

If there is a crossing of arcs (m_{i+1}, n_{i+1}) and (m_k, n_k) in Δ , where $k \neq i$, then $n_k < n_{i+1} < m_k < m_{i+1}$ or $n_{i+1} < n_k < m_{i+1} < m_k$. Consider the case $n_k < n_{i+1} < m_k < m_{i+1}$. Note that $n_k < n_{i+1} < n_i < m_k < m_{i+1}$ and therefore $n_k < n_i < m_k < m_{i+1}$. It follows that there is a crossing of (m_k, n_k) and (m_{i+1}, n_i) in Δ' . In the case $n_{i+1} < n_k < m_{i+1} < m_k$, we have $n_{i+1} < n_i < n_k < m_{i+1} < m_k$ and a crossing of (m_k, n_k) and (m_{i+1}, n_i) in Δ' .

If there is a crossing of arcs (m_i, n_{i+1}) and (m_k, n_k) in Δ' , where $k \neq j$, then $n_k < n_{i+1} < m_k < m_i$ or $n_{i+1} < n_k < m_i < m_k$. Consider the case $n_k < n_{i+1} < m_k < m_i$. Note that $n_k < n_{i+1} < n_i < m_k < m_i$ and therefore $n_k < n_i < m_k < m_i$. It follows that there is a crossing of (m_k, n_k) and (m_i, n_i) in Δ . In the case $n_{i+1} < n_k < m_i < m_k$, we have $n_{i+1} < n_i < n_k < m_i < m_k$ and a crossing of (m_k, n_k) and (m_i, n_i) in Δ .

If there is a crossing of arcs (m_{i+1}, n_i) and (m_k, n_k) in Δ' , where $k \neq j$, then $n_k < n_i < m_k < m_{i+1}$ or $n_i < n_k < m_{i+1} < m_k$. Consider the case $n_k < n_i < m_k < m_{i+1}$. Note that $n_k < n_{i+1} < n_i < m_k < m_{i+1}$ and therefore $n_k < n_{i+1} < m_k < m_{i+1}$. It follows that there is a crossing of (m_k, n_k) and (m_{i+1}, n_{i+1}) in Δ . In the case $n_i < n_k < m_{i+1} < m_k$, we have $n_{i+1} < n_i < n_k < m_{i+1} < m_k$ and a crossing of (m_k, n_k) and (m_{i+1}, n_{i+1}) in Δ . \square

Example: Set $z = 1$, $y = x = 4$ and do the loop (a). We apply our algorithm to the sequence

$$(\infty, 4), (6, 3), (7, 2), (5, 1).$$

For $i = 1$, we compare $(\infty, 4)$ with $(6, 3)$. We have to exchange sources, and we get:

$$(6, 4), (\infty, 3), (7, 2), (5, 1).$$

For $i = 2$, we check $(\infty, 3)$ and $(7, 2)$. Since $\infty > 7 > 3 > 2$ we exchange sources and get:

$$(6, 4), (7, 3), (\infty, 2), (5, 1).$$

For $i = 3$, we compare $(\infty, 2)$ with $(5, 1)$ and (after suitable exchange) we get:

$$(6, 4), (7, 3), (5, 2), (\infty, 1).$$

Now we put $y = 3$ and start the second run of the loop (a). For $i = 1$, we compare $(6, 4)$ and $(7, 3)$. They are in proper positions. For $i = 2$, we compare $(7, 3)$ and $(5, 2)$. We have to exchange sources, and we get:

$$(6, 4), (5, 3), (7, 2), (\infty, 1).$$

We put $y = 2$ and start the third run of the loop (a). If $i = 1$, we check $(6, 4)$ and $(5, 3)$, we exchange sources and get:

$$(5, 4), (6, 3), (7, 2), (\infty, 1).$$

We got the arc-minimal diagram.

Algorithm: Extended bubble sort. Let Δ be an arc diagram with the corresponding sequence of sources and targets

$$(m_1, n_1), \dots, (m_x, n_x)$$

Input: A sequence of sources and targets:

$$(m_1, n_1), \dots, (m_x, n_x).$$

Output: The sequence

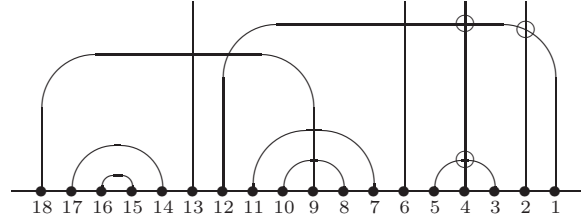
$$(m_{\sigma(1)}, n_1), \dots, (m_{\sigma(x)}, n_x),$$

where $m_{\sigma(1)} \leq m_{\sigma(2)} \leq \dots \leq m_{\sigma(x)}$ and the corresponding arc diagram has no crossings.

Repeat (1)-(8) until there is no crossing in Δ :

1. fix j such that m_j is a minimal in the set $\{m_1, \dots, m_x\}$;
2. consider the sequence $(m_k, n_k), \dots, (m_x, n_x)$, where k is such that n_k is the maximal element in $\{n_1, \dots, n_x\}$ that is less than m_j ;
3. apply to this sequence the bubble sort algorithm;
4. note that we got the sequence $(m_k, n_k), \dots, (m_x, n_x)$, where $m_k = m_j$; note also that the arc (m_k, n_k) has no crossings;
5. remove (m_k, n_k) from the sequence $(m_1, n_1), \dots, (m_x, n_x)$;
6. set $i = i - 1$ for all $i = k + 1, \dots, x$;
7. set $x = x - 1$;
8. come back to (1);

Example: Consider an arc diagram with the following sequence of sources and targets:



$(16, 15), (17, 14), (\infty, 13), (18, 9), (10, 8), (11, 7), (\infty, 6), (\infty, 4), (5, 3), (\infty, 2), (12, 1).$

In the step (1) of the algorithm we have $j = 9$ and $m_j = 5$. Moreover $k = 8$ and $n_k = 4$.

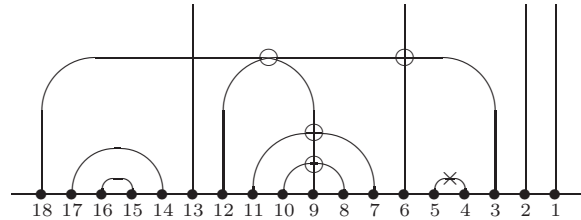
We apply the bubble sort to the four arcs and poles ending at 4, 3, 2, and 1:

$(\infty, 4), (5, 3), (\infty, 2), (12, 1)$

In three steps, the algorithm removes the three encircled intersections. We get:

$(5, 4), (12, 3), (\infty, 2), (\infty, 1).$

We have the following arc diagram:



$(16, 15), (17, 14), (\infty, 13), (18, 9), (10, 8), (11, 7), (\infty, 6), (5, 4), (12, 3), (\infty, 2), (\infty, 1).$

We remove the arc $(5, 4)$ (labelled by an \times) and apply (1)-(8) to the sequence:

$$(16, 15), (17, 14), (\infty, 13), (18, 9), (10, 8), (11, 7), (\infty, 6), (12, 3), (\infty, 2), (\infty, 1).$$

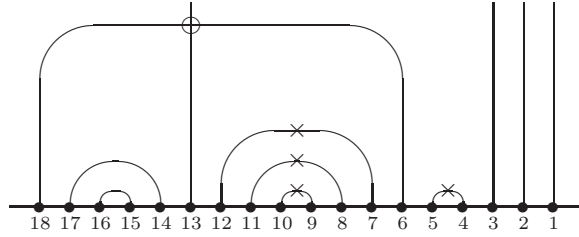
Now we have $j = 5$, $m_j = 10$, $k = 4$ and $n_k = 9$. We apply the bubble sort to the arcs and poles ending at or on the right of 9:

$$(18, 9), (10, 8), (11, 7), (\infty, 6), (12, 3), (\infty, 2), (\infty, 1)$$

In four steps, the algorithm removes the four encircled intersections. We get

$$(10, 9), (11, 8), (12, 7), (18, 6), (\infty, 3), (\infty, 2), (\infty, 1).$$

Our sequence has the form:



$$(16, 15), (17, 14), (\infty, 13), (10, 9), (11, 8), (12, 7), (18, 6), (\infty, 3), (\infty, 2), (\infty, 1).$$

We can remove arcs $(10, 9), (11, 8), (12, 7)$. So we apply (1)-(8) to

$$(16, 15), (17, 14), (\infty, 13), (18, 6), (\infty, 3), (\infty, 2), (\infty, 1).$$

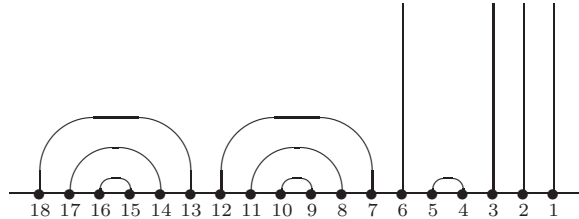
Now $j = 1$, $m_j = 16$, $k = 1$, $n_k = 15$ and we apply the bubble sort to the full sequence:

$$(16, 15), (17, 14), (\infty, 13), (18, 6), (\infty, 3), (\infty, 2), (\infty, 1)$$

The algorithm removes the last intersection in one step. We get

$$(16, 15), (17, 14), (18, 13), (\infty, 6), (\infty, 3), (\infty, 2), (\infty, 1).$$

Our arc diagram has no crossings. The algorithm terminates with the output:



$$(16, 15), (17, 14), (18, 13), (10, 9), (11, 8), (12, 7), (\infty, 6), (5, 4), (\infty, 3), (\infty, 2), (\infty, 1).$$

LEMMA 4.2. *Let Δ be an arc diagram with the corresponding sequence of sources and targets*

$$(m_1, n_1), \dots, (m_x, n_x)$$

and let Δ' be the dominant arc-diagram of the same LR-type. If in Δ there is no multiple poles, then there is a sequence of moves that reduce Δ to Δ' such that after every move the number of crossings is decreasing by one.

Proof. It follows from algorithms and Lemma 4.1. \square

5. Three excursions

5.1. Some projective varieties

We show that projective spaces and Grassmann varieties occur as quotients of diagram varieties.

Denote by $\mathbb{D}_{\alpha, \gamma}^{\beta}(k)$ the subset of the Grassmann variety $\mathbb{G}(|\alpha|, k^{|\beta|})$ consisting of all submodules $U \subseteq N_{\beta}(k)$ such that $U \cong N_{\alpha}(k)$ and $N_{\beta}(k)/U \cong N_{\gamma}(k)$. By $\mathbb{V}_{\alpha, \gamma}^{\beta}(k)$ denote the subset of the affine variety $\mathbb{H}_{\alpha}^{\beta}(k) = \text{Hom}_k(N_{\alpha}(k), N_{\beta}(k))$ consisting of all monomorphisms $f : N_{\alpha}(k) \rightarrow N_{\beta}(k)$ with $\text{Coker } f \cong N_{\gamma}(k)$. The group $\text{Aut}(N_{\alpha}(k))^{op}$ acts freely on $\mathbb{V}_{\alpha, \gamma}^{\beta}(k)$ in the following way. For $\sigma \in \text{Aut}(N_{\alpha}(k))^{op}$ and $f \in \mathbb{V}_{\alpha, \gamma}^{\beta}(k)$ we set

$$\sigma \cdot f = f \circ \sigma.$$

The map

$$F : \mathbb{V}_{\alpha, \gamma}^{\beta}(k) \rightarrow \mathbb{D}_{\alpha, \gamma}^{\beta}(k)$$

defined by $F(f) = (\text{Im } f \subseteq N(\beta))$ is polynomial and its fibers are isomorphic to $\text{Aut}(N_{\alpha}(k))^{op}$.

Remark: Let (1^m) denotes the partition $(1, \dots, 1)$ with m parts.

1. Projective spaces are arc diagram varieties as

$$\mathbb{P}(k^m) = \mathbb{D}_{(1), (1^{m-1})}^{(1^m)}(k) \quad \text{for } m \in \mathbb{N}.$$

Note that $\mathbb{V}_{(1), (1^{m-1})}^{(1^m)}(k) = k^m \setminus \{0\}$.

2. Grassmann varieties can be realized as

$$\mathbb{G}(\ell, k^m) = \mathbb{D}_{(1^{\ell}), (1^{m-\ell})}^{(1^m)}(k) \quad \text{for } \ell, m \in \mathbb{N}, \ell \leq m.$$

The variety $\mathbb{V}_{(1^{\ell}), (1^{m-\ell})}^{(1^m)}(k)$ is consisted of all $\ell \times m$ matrices with maximal rank.

For finite fields, the size of the projective varieties is under control:

We have

$$|\mathbb{D}_{\alpha,\gamma}^\beta(\mathbb{F}_p)| = g_{\alpha,\gamma}^\beta(p)$$

and

$$|\mathbb{V}_{\alpha,\gamma}^\beta(\mathbb{F}_p)| = |\text{Aut} N_\alpha(\mathbb{F}_p)| \cdot |\mathbb{D}_{\alpha,\gamma}^\beta(\mathbb{F}_p)| = |\text{Aut} N_\alpha(\mathbb{F}_p)| \cdot g_{\alpha,\gamma}^\beta(p).$$

5.2. Degenerations of nilpotent operators

Classical Hall polynomials allow to investigate geometric properties of nilpotent operators.

Let k be an arbitrary algebraically closed field. We consider the affine variety $\mathbb{M}_n(k)$ consisting of all $n \times n$ -matrices with coefficients in k . On $\mathbb{M}_n(k)$ we consider the Zariski topology and on all subsets of $\mathbb{M}_n(k)$ we work with the induced topology. By $\mathbb{M}_n^0(k)$ denote the closed subset of $\mathbb{M}_n(k)$ consisted of nilpotent matrices. On $\mathbb{M}_n^0(k)$ acts the general linear group $GL_n(k)$ via conjugations $g \cdot A = gAg^{-1}$. Orbits of this action correspond bijectively to isomorphism classes of objects in $\mathcal{N}(k, n)$, where $\mathcal{N}(k, n)$ is the full subcategory of $\mathcal{N}(k)$ consisting of all objects $N_\alpha = N_\alpha(k)$ such that $\dim_k N_\alpha = n$. Denote by $G_\alpha = G_\alpha(k)$ the orbit of N_α in $\mathbb{M}_n^0(k)$.

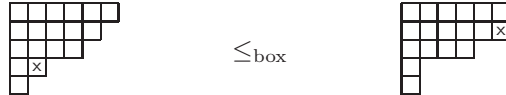
Definition: Let N_α and N_β be objects in $\mathcal{N}(k, n)$. The relation $N_\alpha \leq_{\text{deg}} N_\beta$ holds if $G_\beta(k) \subseteq \overline{G_\alpha(k)}$ in $\mathbb{M}_n^0(k)$, where $\overline{G(k)}$ is the closure of $G(k)$.

The following theorem is well known (see [12, I.3])

THEOREM 5.1. *Let N_α and N_β be objects in $\mathcal{N}(k, n)$. The relation $N_\alpha \leq_{\text{deg}} N_\beta$ holds if and only if $\sum_{i=1}^m \alpha'_i \leq \sum_{i=1}^m \beta'_i$ for all $m \in \mathbb{N}$, where α' denotes the conjugate partition of α .*

Let α and β be partitions of n . We write $\alpha \mapsto_{\text{box}} \beta$ if there exists $i < j$ such that $\alpha_i = \beta_i - 1$, $\alpha_j = \beta_j + 1$ and $\alpha_k = \beta_k$ for $k \neq i, j$. We define the **box order** $\alpha \leq_{\text{box}} \beta$ to be the partial order generated by all moves \mapsto_{box} .

If we look at Young diagrams, the box order is generated by a sequence of moves of type (going up with a box):



THEOREM 5.2. *Let α and β be partitions of n . Then*

$$N_\alpha \leq_{\text{deg}} N_\beta \quad \text{if and only if} \quad \alpha \leq_{\text{box}} \beta.$$

Proof. Let $N_\alpha \leq_{\text{deg}} N_\beta$ and $\alpha \neq \beta$. It follows that $\sum_{i=1}^m \alpha'_i \leq \sum_{i=1}^m \beta'_i$ for all $m \in \mathbb{N}$. Let s be the minimal natural number such that $\sum_{i=1}^s \alpha'_i < \sum_{i=1}^s \beta'_i$. It follows that $\alpha'_i = \beta'_i$ for all $i = 1, \dots, s-1$ and $\alpha'_s < \beta'_s$. Since $\sum_{i=1}^\infty \alpha'_i =$

$\sum_{i=1}^{\infty} \beta'_i = n$, there exists $t > s$ such that $\alpha'_t > \beta'_t$. Chose t minimal with this property. Let γ be the partition such that

$$\gamma' = (\beta'_1, \dots, \beta'_{s-1}, \beta'_s - 1, \beta'_{s+1}, \dots, \beta'_{t-1}, \beta'_t + 1, \beta'_{t+1}, \dots).$$

It is straightforward to check that $\gamma \leq_{\text{box}} \beta$ and

$$\sum_{i=1}^m \alpha'_i \leq \sum_{i=1}^m \gamma'_i \leq \sum_{i=1}^m \beta'_i,$$

for all m , and

$$\sum_{i=1}^s \alpha'_i \leq \sum_{i=1}^s \gamma'_i < \sum_{i=1}^s \beta'_i.$$

Therefore we have $N_\alpha \leq_{\text{deg}} N_\gamma$. Continuing this procedure we prove that $\alpha \leq_{\text{box}} \beta$.

Conversely, assume that $\alpha \leq_{\text{box}} \beta$ is given by single "box move". It is easy to prove that $\sum_{i=1}^m \alpha'_i \leq \sum_{i=1}^m \beta'_i$ for all $m \in \mathbb{N}$. Therefore $N_\alpha \leq_{\text{deg}} N_\beta$ and we are done. \square

Combining results presented in [12, I.3] and in [14] we can prove the following.

THEOREM 5.3. *Let*

$$0 \rightarrow N_\alpha \rightarrow N_\lambda \rightarrow N_\beta \rightarrow 0,$$

$$0 \rightarrow N_\alpha \rightarrow N_\gamma \rightarrow N_\beta \rightarrow 0$$

be short exact sequences of $k[T]$ -modules. If $N_\lambda \leq_{\text{deg}} N_\gamma$, then

$$\deg g_{\alpha\beta}^\lambda \leq \deg g_{\alpha\beta}^\gamma.$$

Proof. Assume that $N_\lambda \leq_{\text{deg}} N_\gamma$. By [12, I.3], for any $m \geq 1$:

$$\sum_{i=1}^m \lambda'_i \leq \sum_{i=1}^m \gamma'_i.$$

It follows from [14, Section I, 1.11] that for any $m \geq 1$:

$$\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m \gamma_i.$$

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\gamma = (\gamma_1, \dots, \gamma_n)$. Since $\lambda_1 + \dots + \lambda_k = \gamma_1 + \dots + \gamma_n$ and $\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m \gamma_i$, we have $n \geq k$. We prove that $n(\lambda) \leq n(\gamma)$. Consider the equality $k \cdot \lambda_1 + \dots + k \cdot \lambda_k = k \cdot \gamma_1 + \dots + k \cdot \gamma_n$ and subtract inequalities $\sum_{i=1}^m \lambda_i \geq \sum_{i=1}^m \gamma_i$, for $m = 1, \dots, k$. We get

$$\begin{aligned} n(\lambda) &= 0 \cdot \lambda_1 + 1 \cdot \lambda_2 + \dots + (k-1) \cdot \lambda_k \\ &\leq 0 \cdot \gamma_1 + 1 \cdot \lambda_2 + \dots + (k-1) \cdot \gamma_k + k \cdot \gamma_{k+1} + \dots + k \cdot \gamma_n \leq n(\gamma). \end{aligned}$$

This finishes the proof, because

$$\deg g_{\alpha\beta}^\lambda = n(\lambda) - n(\alpha) - n(\beta) \leq n(\gamma) - n(\alpha) - n(\beta) = \deg g_{\alpha\beta}^\gamma.$$

\square

Connections of Young tableaux and partitions with degenerations and generic extensions of nilpotent operators are also studied in [10].

5.3. A partial ordering on LR-tableaux

Let Γ be an LR-tableau of type (α, β, γ) . Note that the poset structure on \mathcal{D}_Γ is just the restriction of $\mathcal{D}_{\alpha, \gamma}^\beta$ to the arc diagrams in Γ . By “identifying” those arc diagrams we obtain a poset structure $\overline{\mathcal{D}}_{\alpha, \gamma}^\beta$ on the set of LR-tableaux of type (α, β, γ) ; the relation is given by

$$\Gamma \leq \Gamma' \iff \mathbb{V}_{\Gamma'} \cap \overline{\mathbb{V}}_\Gamma \neq \emptyset.$$

We can characterize this order relation in different ways.

Assume that $\alpha_1 \leq 2$ holds, then an LR-tableau Γ of type (α, β, γ) is given by three partitions $\gamma \subset \tilde{\gamma} \subset \beta$ where the **intermediate partition** $\tilde{\gamma}$ is such that $\beta \setminus \tilde{\gamma}$ consists of all boxes $\boxed{1}$ and $\tilde{\gamma} \setminus \gamma$ consists of all boxes $\boxed{2}$.

PROPOSITION 5.4. *The following assertions are equivalent for LR-tableaux Γ, Γ' of type (α, β, γ) .*

1. $\Gamma \leq \Gamma'$.
2. There exists $\Delta' \in \mathcal{D}_{\Gamma'}$ such that $\mathbb{V}_{\Delta'} \subset \overline{\mathbb{V}}_\Gamma$.
3. There are $\Delta \in \mathcal{D}_\Gamma$ and $\Delta' \in \mathcal{D}_{\Gamma'}$ such that $\Delta \leq_{\text{arc}} \Delta'$.
4. The intermediate partitions $\tilde{\gamma}$ for Γ and $\tilde{\gamma}'$ for Γ' satisfy $\tilde{\gamma}' \leq_{\text{box}} \tilde{\gamma}$.

Proof. The equivalence of 1. and 2. is clear from the definitions. To see that 2. implies 3. note that there is an arc diagram Δ such that $\overline{\mathbb{V}}_\Gamma = \overline{\mathbb{V}}_\Delta$; the converse holds since $\mathbb{V}_\Delta \subset \mathbb{V}_\Gamma$ implies that $\overline{\mathbb{V}}_\Delta \subset \overline{\mathbb{V}}_\Gamma$.

We show that 3. implies 4. Suppose $\Delta \leq_{\text{arc}} \Delta'$, then there is a sequence of moves which convert Δ to Δ' . Note that moves of type (A) or (B) leave the underlying LR-tableau unchanged, while moves of type (C) or (D) exchange the positions of a box $\boxed{1}$ with a box $\boxed{2}$. The direction is order-reversing: If $\Delta \leq_{\text{arc}} \Delta'$ then $\tilde{\gamma}' \leq_{\text{box}} \tilde{\gamma}$.

For the converse we assume that the intermediate partitions $\tilde{\gamma}$ for Γ and $\tilde{\gamma}'$ for Γ' satisfy the relation $\tilde{\gamma}' \leq_{\text{box}} \tilde{\gamma}$ and are such that $\tilde{\gamma}$ is obtained from $\tilde{\gamma}'$ by the move of a single box, say from the a -th row up into the b -th row. Let Δ' be the unique arc diagram of type Γ' with the maximal number of intersections. It follows that there is an arc in Δ' starting at a which intersects an arc or pole in Δ' ending at b . The arc move of type (C) or (D) which resolves this intersection yields a diagram Δ of type Γ . Thus, $\Delta \leq_{\text{arc}} \Delta'$. \square

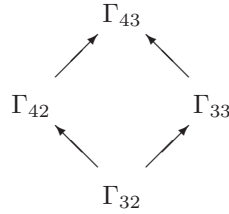
PROPOSITION 5.5. 1. *The poset $\overline{\mathcal{D}}_{\alpha, \gamma}^\beta$ has a unique maximal element, it is the LR-tableau given by the unique arc diagram with the maximal number of intersections. Equivalently, it is the LR-tableau of type (α, β, γ) in which the boxes $\boxed{2}$ are in the largest available rows.*

2. The poset $\overline{\mathcal{D}}_{\alpha,\gamma}^\beta$ has a unique minimal element, it is given by the unique LR-tableau that can be refined only to arc diagrams with no intersections. Equivalently, it is the LR-tableau of type (α, β, γ) in which the boxes $\boxed{2}$ are in the smallest available rows.

Proof. The first statement follows from Theorem 3.1 and Proposition 5.4. Namely, if Δ is the unique arc diagram with a maximum number of intersections, then \mathbb{V}_Δ is contained in the closure of any other stratum. Recall from [11, Proof of Theorem 5.7] that this LR-tableau is such that the entries $\boxed{2}$ are in the largest available rows.

Consider the LR-tableau Γ which is such that the entries $\boxed{2}$ are in the smallest available rows (to obtain Γ , proceed rowwise from the top, and put in each row the largest possible number of $\boxed{2}$'s). Let Δ be an arc diagram of type Γ . It is not possible to resolve an intersection in Δ by an arc move of type (C) or (D) since each such move lifts a box $\boxed{2}$ into a higher row. Since moves of type (A) and (C), and of type (B) and (D) occur pairwise, it is not possible to resolve an intersection in Δ by any arc moves, i.e. Δ has no intersection. \square

Example: We revisit Example 3.1. Corresponding to the partitions $\alpha = (2, 2, 1, 1)$, $\beta = (4, 3, 3, 2, 2, 1)$, $\gamma = (3, 2, 2, 1, 1)$ are the four LR-tableaux Γ_{43} , Γ_{42} , Γ_{33} and Γ_{32} pictured in Section 2. Here is the Hasse diagram for the partial ordering in $\overline{\mathcal{D}}_{\alpha,\gamma}^\beta$.



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